Self-organized and driven phase synchronization in coupled map scale free networks

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As a model of evolving networks, we study coupled logistic maps on scale free networks. For small coupling strengths nodes show turbulent behavior but form phase synchronized clusters as coupling increases. We identify two different ways of cluster formation. For small coupling strengths we get *self-organized clusters* which have mostly intra-cluster couplings and for large coupling strengths there is a crossover and reorganization to *driven clusters* which have mostly inter-cluster couplings. In the novel driven synchronization the nodes of one cluster are driven by those of the others.

Recently, there is considerable interest in complex systems described by networks or graphs with complex topology [1]. One significant recent discovery in the field of complex networks is the observation that a number of naturally occurring large and complex networks are scale free, i.e. the probability that a node in the network is connected to k other nodes decays as a power law. These networks are found in many diverse systems such as the nervous systems [2], social groups [3], world wide web [4], metabolic networks [5], food webs [6] and citation network [7]. Barabasi and Albert [8] have given a simple growth model based on preferential attachments for the scale free networks.

Most networks in the real world consist of dynamical elements interacting with each other. Thus in order to understand properties of such networks, we study a coupled map model of scale free networks. Over the past decade, much work on coupled maps has used regular coupling schemes. They show rich phenomenology arising when opposing tendencies compete; the nonlinear dynamics of the maps which in the chaotic regime tends to separate the orbits of different elements, and the couplings that tend to synchronize them. Coupled map lattices with nearest neighbor or short range interactions show interesting spatio-temporal patterns, and intermittent behaviour [10]. Globally coupled maps (GCM) where each node is connected with all other nodes, show interesting synchronized behaviour [11]. Random networks with large number of connections also show synchronized behaviour for large coupling strengths. [12].

In this letter, we study the dynamics of coupled maps on scale free networks. In particular, we investigate the clustering and synchronization properties of such dynamically evolving networks. We find that as the network evolves, it splits into several phase synchronized clusters. Phase synchronization is obviously because of the couplings between the nodes of the network and may be achieved in two different ways. (i) The nodes of a cluster can be synchronized because of intra-cluster couplings. We refer to this as the *self-organized synchronization.* (ii) Alternately, the nodes of a cluster can be synchronized because of inter-cluster couplings. We refer to this as the *driven synchronization.* We find examples of both these types of phase synchronized clusters in scale free networks with a crossover and reorganization of nodes between the two types as the coupling strength is varied. For small couplings synchronization is of the self-organized type while for large couplings it is of the driven type. We will discuss several examples of such clusters in natural systems afterwards.

Consider a network of N nodes that are coupled with each other through connections of the scale free type. Let each node of the network be assigned a dynamical variable $x^i, i = 1, 2, \ldots, N$. The evolution of the dynamical variables is given by

$$x^i_{t+1} = (1-\epsilon) f(x^i_t) + \frac{\epsilon}{\sum_j C_{ij}} \sum C_{ij} g(x^j_t) \tag{1}$$

where x^i_n is the dynamical variable of the i-th node ($1 \le i \le N$) at the $t-th$ time step, C is the coupling matrix with elements C_{ij} taking values 1 or 0 depending upon whether i and j are connected or not. Note that C is a symmetric matrix with diagonal elements zero. The function $f(x)$ defines the local nonlinear map and the function $g(x)$ defines the nature of coupling between the nodes. In this paper, we define the local dynamics by the logistic map, $f(x) = \mu x(1-x)$. The coupling function is taken as the identity mapping, $g(x) = x$.

The scale free network of N nodes is generated by using the model of Barabasi et.al. [9]. Starting with a small number, m_0, of nodes, at every time step a new node with $m \le m_0$ connections is added. The probability $\pi(k_i)$ that a connection starting from this new node is connected to a node i depends on the degree k_i of node i (preferential attachment) and is given by $\pi(k_i) = (k_i+1)/(\sum_j (k_j+1))$. After τ time steps the model leads to a random network

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with $N = \tau + m_0$ nodes and $m\tau$ connections. This model leads to a scale free network, i.e. the probability $P(k)$ that a node has a degree k decays as a power law, $P(k) \sim k^{\lambda}$ where λ is a constant. For the type of probability law $\pi(k)$ that we have used $\lambda = 3$. Other forms for the probability $\pi(k)$ are possible which give different values of λ. However, the results reported in this letter do not depend on the exact form of $\pi(k)$ except that it should lead to a scale free network.

Synchronization of coupled dynamical systems may be defined in various ways. The perfect synchronization corresponds to the dynamical variables for different nodes having identical values. The phase synchronization corresponds to the dynamical variables for different nodes having values with some definite relations [13]. In scale free networks, we find that when the local dynamics of the nodes (i.e. function $f(x)$) is in the chaotic zone, perfect synchronization is obtained only for clusters with small number of nodes. However, interesting results are obtained when we study phase synchronized behaviour. For our study we define the phase synchronization as follows [14]. Let n_i and n_j denote the number of times the dynamical variables x_t^i and x_t^j, $t = 1, 2, \ldots, T$ for the nodes i and j show local minima during the time interval T. Let n_{ij} denote the number of times these local minima match with each other. We define the phase distance between the nodes i and j as $d_{ij} = 1 - 2n_{ij}/(n_i + n_j)$. Clearly, $d_{ij} = 0$ when all minima of variables x^i and x^j match with each other and $d_{ij} = 1$ when none of the minima match. We say that nodes i and j are phase synchronized if $d_{ij} = 0$. Also, a cluster of nodes is phase synchronized if all pairs of nodes belonging to that cluster are phase synchronized.

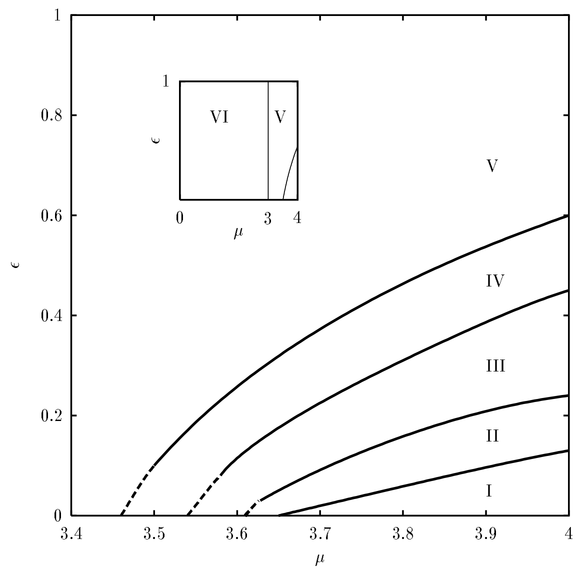

FIG. 1. Phase diagram showing turbulent, phase synchronized and coherent regions in the two parameter space of μ and ϵ. Different regions are I. Turbulent region, II. Mixed region, III. Partially ordered region, IV. Ordered quasiperiodic region, V. Ordered periodic region, VI. Coherent region. Calculations are for $N = 50, m = 1, T = 100$. Region boundaries are determined based on changes in the behaviour of the largest Lyapunov exponent (see Fig. (2)) and observing the asymptotic behaviour using several initial conditions. The dashed lines indicate uncertainties in determining the boundaries. The inset shows the phase diagram for the entire range of parameter μ i.e. from 0 to 4.

We now present numerical results of our model. We generate scale free networks using the algorithm defined above and then study coupled dynamics of variables associated with nodes of the network. Starting from random initial conditions the dynamics of Eq. (1), after an initial transient, leads to interesting phase synchronized behaviour. Fig. (1) shows the phase diagram in the two parameter space defined by μ and ϵ for $m = 1, N = 50, T = 100$. To understand the phase diagram let us first consider different states which are obtained by the coupled dynamics.
(a) Coherent state: Nodes form a single synchronized cluster.
(b) Ordered state: Nodes form two or more clusters.
(c) Partially ordered states: Nodes form a few clusters with some nodes not attached to any clusters.
(d) Turbulent state: All nodes behave chaotically with no cluster formation.

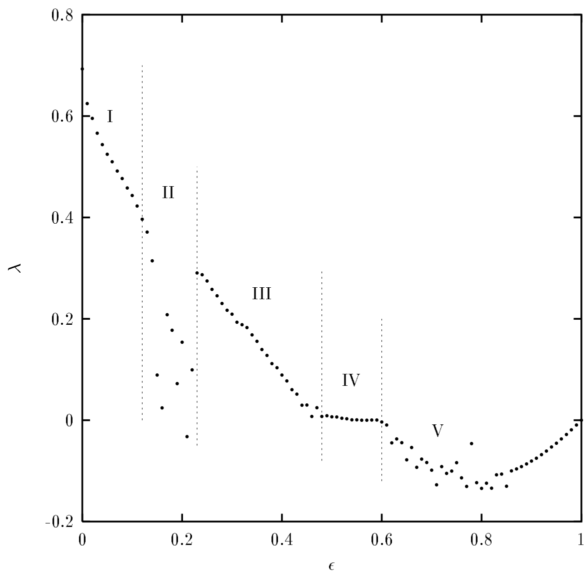

FIG. 2. Largest Lyapunov exponent, λ, is plotted as a function of ϵ for $\mu = 4.0$. Different regions are labeled as in Fig. (1).

For $\mu < 3$, we get a stable coherent region (region VI) with all nodes having the fixed point value. To understand the remaining phase diagram, consider the line $\mu = 4$. Fig. (2) shows the largest Lyapunov exponent λ as a function of the coupling strength ϵ for $\mu = 4$. We can identify five different regions as ϵ increases from 0 to 1; namely turbulent region, mixed region, partially ordered region, ordered quasiperiodic region and ordered periodic region as shown by regions I to V in Figs. (1) and (2). For small values of ϵ, we observe a turbulent behaviour with all nodes evolving chaotically and there is no phase synchronization. As ϵ increases further we get into a mixed region (region II) which shows variety of phase synchronized behaviour, namely ordered chaotic, ordered quasiperiodic, ordered periodic and partially ordered, depending on the initial conditions. Next region (region III) shows partially ordered chaotic behaviour. Here, the number of clusters as well as the number of nodes in the clusters depend on the initial conditions and also they change with time. There are several isolated nodes not belonging to any cluster. Many of these nodes are of the floating type which keep on switching intermittently between independent evolution and phase synchronized evolution attached to some cluster. Last two regions are ordered quasiperiodic and ordered periodic regions. In these regions, the network always splits into two clusters. The two clusters are perfectly anti-phase synchronized with each other, i.e. when the nodes belonging to one cluster show minima those belonging to the other cluster show maxima.

We now investigate the nature of phase ordering in different regions of the phase diagram. We first concentrate on the middle of regions II and V where we observe ordered periodic behaviour. In both cases the largest Lyapunov exponent is negative. In Fig. (3a) and (3b) we show the coupling matrix C (solid circles) and the nodes belonging to the two clusters in regions II and V respectively. In Fig. (3a) we observe that there are no inter-cluster couplings between the nodes of the two clusters except one coupling i.e. all the couplings except one are of the intra-cluster type. The phase synchronization in this case is clearly of the *self-organized* type. Exactly opposite behaviour is observed for the region V (Fig. (3b)). Here, we find that all the couplings are of inter-cluster type with no intra-cluster couplings. This is clearly the phenomena of *driven synchronization* where nodes of one cluster are driven into a phase synchronized state due to the couplings with nodes of the other cluster. The phenomena of driven synchronization in this region is a very robust one in the sense that it is obtained for almost all initial conditions, the transient time is very small, the nodes belonging to the two clusters are uniquely determined and we get a stable solution.

We observe that for small values of ϵ the self organized behaviour dominates while for large ϵ driven behaviour dominates. As the coupling parameter ϵ increases from zero and we enter region II, we observe phase synchronized clusters of the self organized type. Region III acts as a crossover region from the self-organized to the driven behaviour. Here, the clusters are of mixed type. The number of inter-cluster couplings is approximately same as the number of intra-cluster couplings. In this region there is a competition between the self-organized and driven behaviour. This appears to be the reason for the formation of several clusters and floating nodes as well as the sensitivity of these to the initial conditions. As ϵ increases, we get into region IV where the driven synchronization dominates and most of the connections between the nodes are of the inter-cluster type. This driven synchronization is further stabilized in region V with two perfectly anti-phase synchronized driven clusters.

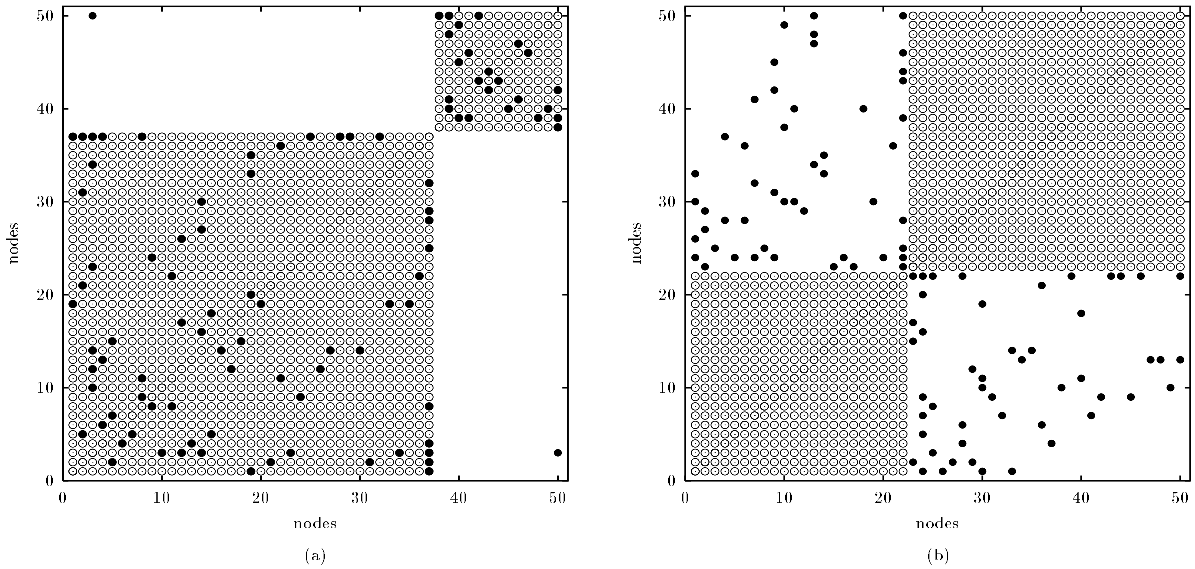

FIG. 3. Figures (a) and (b) show formation of phase synchronized clusters for $\mu = 4$ and $\epsilon = 0.16$ and 0.8 respectively. For clarity of display, the nodes are renumbered in each case such that nodes belonging to the same cluster have sequential numbers. The nodes of clusters are shown by open circles in squares. The elements of the coupling matrix are shown by solid circles.

Geometrically, the organization of the network into connections of both self-organized and driven types is always possible for $m = 1$. For $m = 1$, our growth algorithm generates a tree type structure. A tree can always be broken into two or more disjoint clusters with only intra-cluster couplings by breaking one or more connections. Clearly, this splitting is not unique. A tree can also be divided into two clusters by putting connected nodes into different clusters. This division is unique and leads to two clusters with only inter-cluster couplings.

For $m > 1$ the dynamics of Eq. (1) leads to a similar phase diagram as in Fig (1) with region II dominated by self-organized synchronization and regions IV and V dominated by driven synchronization. Though perfect inter- and intra-cluster couplings between the nodes as displayed in Figs. (3a) and (3b) are no longer observed, clustering in region II is such that most of the couplings are of intra-cluster type while for regions IV and V they are of the inter-cluster type. As m increases the regions I and II are mostly unaffected, but regions IV and V shrink while region III grows in size.

The phenomena of self-organized and driven behaviour persists for the largest size network that we have studied ($N = 1000$). The mixed region showing self-organized behaviour is mostly unaffected while the ordered regions showing driven behaviour show a small shrinking in size.

We have generated the networks using the preferential attachment law for $\pi(k_i)$ leading to a power law distribution of the degree of nodes. If instead we use a non-preferential law of attachment i.e. the probability of connecting to any node does not depend on the degree of that node but is a constant, then the distribution of degree of nodes shows an exponential dependence rather than a power law. In this case we do not observe any phase synchronized behaviour. Thus the scale free nature appears to be important for observing the phase synchronized behaviour.

There are several examples of self-organized and driven behaviour in naturally occurring systems. Self-organized behaviour is more common and is easily observed. Examples are social, ethnic and religious groups, political groups, cartel of industries and countries, herds of animals and flocks of birds, different dynamic transitions such as self-organized criticality etc. The driven behaviour is not so common. An interesting example is the behaviour of fans during a match between traditional rivals. Before the match the fans may act as individuals (turbulent behavior) or form self-organized clusters such as a single cluster of fans of the game or several clusters of fans of different star players. During the match there can be a crossover to a driven behaviour. When the match reaches a feverish pitch, i.e. the strength of the interaction increases, the fans are likely to form two phase synchronized groups. The response of the two groups will be anti-phase synchronized with each other. Another example of crossover to a driven behaviour is the conflict in Bosnia where a society organized into villages and towns was split into ethnic groups.

As discussed in the introduction several naturally occurring networks show scale free behaviour and it is likely to be a generic behaviour of several naturally growing networks. We expect to observe both the self-organized and

driven synchronization behaviour reported in this paper in such systems. Network properties of many examples of self-organized and driven behaviour discussed above are not known and it is possible that some of these examples may have scale free type of networks.

To conclude we have found interesting self-organized and driven phase synchronization behaviour in coupled maps on scale free networks. Self-organized synchronization is characterized by dominant intra-cluster couplings and is found when strength of the couplings is small as compared to the local dynamics. As the coupling strength increases there is a crossover from the self-organized to the driven behaviour which also involves reorganization of nodes into different clusters. The driven behaviour is characterized by inter-cluster couplings and is found when strength of the couplings is large as compared to the local dynamics.

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